Minimal Number of Generators and Minimum Order of a Non-Abelian Group whose Elements Commute with Their Endomorphic Images

A. Abdollahi * **A. Faghihi** and **A. Mohammadi Hassanabadi**
Department of Mathematics,
University of Isfahan,
Isfahan 81746-73441,
Iran.

ABSTRACT. A group in which every element commutes with its endomorphic images is called an E-group. If p is a prime number, a p-group G which is an E-group is called a pE-group. Every abelian group is obviously an E-group. We prove that every 2-generator E-group is abelian and that all 3-generator E-groups are nilpotent of class at most 2. It is also proved that every infinite 3-generator E-group is abelian. We conjecture that every finite 3-generator E-group should be abelian. Moreover we show that the minimum order of a non-abelian pE-group is p^8 for any odd prime number p and this order is 2^7 for $p=2$. Some of these results are proved for a class wider than the class of E-groups.

1. Introduction and results

A group in which each element commutes with its endomorphic images is called an "E-group". It is well-known (see e.g., [8]) that a group G is an E-group if and only if the near-ring generated by the endomorphisms of G in the near-ring of maps on G is a ring.

Since in an E-group every element commutes with its image under inner automorphisms, every E-group is a 2-Engel group, and so they are nilpotent of class at most 3 (see [6], or [11, Theorem 12.3.6]). Throughout the paper p denotes a prime number. We call an E-group which is also a p-group, a pE-group. Since a finite E-group can be written as a direct product of its Sylow subgroups, and any direct factor of an E-group is an E-group [9], so we need only consider pE-groups.

The first examples of non-abelian pE-groups are due to R. Faudree [3], which are defined as follows:

$$G = \langle a_1, a_2, a_3, a_4 \mid a_i^{p^2} = 1, [a_i, a_j, a_k] = 1, \;\; i, j, k \in \{1, 2, 3, 4\}$$
$$[a_1, a_2] = a_1^p, [a_1, a_3] = a_3^p, [a_1, a_4] = a_4^p, [a_2, a_3] = a_2^p, [a_2, a_4] = 1, [a_3, a_4] = a_3^p \;\rangle,$$

1991 *Mathematics Subject Classification.* 20D45; 20E36.

Key words and phrases. 2-Engel groups, p-groups, endomorphisms of groups, near-rings.

* Corresponding Author. e-mail: a.abdollahi@math.ui.ac.ir.

This work was supported partially by the Center of Excellence for Mathematics, University of Isfahan.

where p is any odd prime number. Note that the above example is false for $p = 2$. This is because, when $p = 2$ then the map α defined by

$$a_1^\alpha = a_1^{-1}a_2a_4,\ a_2^\alpha = a_3,\ a_3^\alpha = a_4,\ a_4^\alpha = a_1a_4$$

can be extended to an endomorphism of G. But $[a_3^\alpha, a_3] = [a_4, a_3] \neq 1$, so that G is not an E-group.
All known examples of non-abelian E-groups have nilpotency class 2 (see [1], [2], [3], [7]). In this paper, we see new examples of E-groups.
A. Caranti posed the question [5, Problem 11.46 a] of whether there exists a finite $3E$-group of nilpotency class 3. (Note that every E-group without elements of order 3 is nilpotent of class at most 2).
Some partial (negative) answers to this question are as follows: finite $3E$-groups of exponent dividing 9 are nilpotent of class at most 2 [9]; every 2-generator E-group is nilpotent of class at most 2 (since they are 2-Engel). Here we concentrate on the following questions:
(1) What is the least number of generators of a finitely generated non-abelian E-group?
(2) What is the minimum order of a finite non-abelian pE-group?
We prove

Theorem 1.1. *Every finite 3-generator E-group is nilpotent of class at most 2.*

Theorem 1.2. *Let G be a $3E$-group. If $|G| \leq 3^{10}$, then G is nilpotent of class at most 2.*

As we said, it is easily seen that every 2-generator E-group is nilpotent of class at most 2. In fact an stronger result holds, namely:

Theorem 1.3. (i) *Every 2-generator group is abelian if and only if it is an E-group.*
(ii) *Every infinite 3-generator group is abelian if and only it is an E-group.*

Thus, by Theorem 1.3 and Faudree's examples of E-groups, the minimal number of generators of a non-abelian E-group is 3 or 4. We conjecture that this minimal number must be 4, that is every 3-generator E-group is abelian.
In response to question (2), we prove

Theorem 1.4. *For any prime number p, every pE-group of order at most p^6 is abelian.*

Theorem 1.5. (i) *For any odd prime number p, every pE-group of order at most p^7 is abelian.*
(ii) *There exist non-abelian $2E$-groups of order 2^7.*

As a result of Theorems 1.4 and 1.5 and Faudree's examples of E-groups, we conclude that the minimum order of a finite non-abelian pE-group is p^8, for any odd prime number p and this order is 2^7 for $p = 2$.

2. **Preliminary definitions and results**

Let G be a group, H be an element or a subgroup of G and n be a positive integer. We denote by G', $\Phi(G), G^n$, $\gamma_3(G)$, $Z(G)$, $Z_2(G)$, $\exp(G)$, $Aut(G)$, $End(G)$, $C_G(H)$, Q_8, and C_n, respectively the derived

subgroup, the Frattini subgroup, the subgroup generated by n-powers of the elements, the third term of the lower central series, the center, the second center, the exponent, the automorphism group, the set of endomorphisms of G, the centralizer of H in G, the quaternion group of order 8, and the cyclic group of order n. For a finite p-group G and a positive integer n, we denote by $\Omega_n(G)$ the subgroup generated by elements x such that $x^{p^n} = 1$. If G is a nilpotent group, we denote by $\mathrm{cl}(G)$ the nilpotency class of G. If G is a finite group, $d(G)$ will denote the minimum number of generators of G. If a, b and c are elements of a group, we denote by $[a, b]$ the commutator $a^{-1}b^{-1}ab$ and we define $[a, b, c] = [[a, b], c]$, as usual. An automorphism α of a group G is called central if $x^{-1}x^{\alpha} \in Z(G)$ for all $x \in G$.
As we have found that some of our results are valid for a class of finite p-groups larger than pE-groups, we introduce the following class of p-groups for every prime number p:

Definition 2.1. *A finite p-group G is called a $p\mathcal{E}$-group if G is a 2-Engel group and there exists a non-negative integer r such that $\Omega_r(G) \leq Z(G)$ and $\exp(\frac{G}{G'}) = p^r$.*

Remark 2.2. We know that a finite pE-group is a $p\mathcal{E}$-group [9]; but the converse is false in general as one can see that Q_8 and the group

$$G = \langle a_1, a_2, a_3 : [a_i, a_j, a_k] = 1, (i, j, k \in \{1, 2, 3\}), a_1^4 = 1, a_1^2 = a_2^2 = a_3^2 = [a_1, a_2], [a_2, a_3]^2 = [a_1, a_3]^2 = 1\rangle,$$

are $2\mathcal{E}$-groups which are not $2E$-groups. The quaternion group

$$Q_8 = \langle a, b \mid a^b = b^{-1}, a^2 = b^2, a^4 = 1\rangle$$

is not an E-group, since $[a, b] \neq 1$ and the map α which sends a to b and b to a, can be extended to an automorphism of Q_8.

Lemma 2.3. *If G is a finite 2-Engel p-group, $p > 2$ and $m \in \mathbb{N}$, then $G^{p^m} = \{g^{p^m} : g \in G\}$ and G is regular.*

Proof. This follows easily from the following fact which is valid for all $a, b \in G$:

$$a^{p^m} b^{p^m} = (ab)^{p^m} [a, b]^{\frac{p^m(p^m-1)}{2}} = \left(ab[a, b]^{\frac{p^m-1}{2}}\right)^{p^m}.$$

□

Lemma 2.4. *Let G be a $p\mathcal{E}$-group such that $\exp(\frac{G}{G'}) = p^r$.*
(i) $\exp(G') = \exp(G/Z(G))$ *and* $\exp(G) = p^r \exp(G')$.
(ii) *if* $\mathrm{cl}(G) = 2$, *then* $\exp(G') \leq p^r$.
(iii) *if* $\mathrm{cl}(G) = 3$, *then* $p = 3$ *and* $\exp(G') = 3^{r+1}$.

Proof. (i) We have

$$\begin{aligned} \exp(G/Z(G)) \text{ divides } n &\Leftrightarrow [a^n, b] = 1 \ \forall\, a, b \in G \\ &\Leftrightarrow [a, b]^n = 1 \ \forall\, a, b \in G \\ &\Leftrightarrow \exp(G') \text{ divides } n. \end{aligned}$$

This shows that $\exp(G') = \exp(G/Z(G))$. For the second part of (i), note that if $\exp(G') = p^t$, then $G^{p^{r+t}} \leq (G^{p^r})^{p^t} \leq (G')^{p^t} = 1$. Also if $G^{p^{r+t-1}} = 1$, then $G^{p^{t-1}} \leq \Omega_r(G) \leq Z(G)$ and so $\exp(G') \leq p^{t-1}$, which is impossible. It follows that $\exp(G) = p^{r+t}$.
(ii) This follows from (i) and the fact that $G' \leq Z(G)$.
(iii) Note that in every 2-Engel group K, $\gamma_3(K)^3 = 1$ (see [11, Theorem 12.3.6]). Thus, since $\mathrm{cl}(G) = 3$, we have $p = 3$ and one can write

$$(G')^{3^{r+1}} = [G^{3^r}, G]^3 \leq [G', G]^3 = \gamma_3(G)^3 = 1$$

which gives $\exp(G') \leq 3^{r+1}$. If $\exp(G') \leq 3^r$, then $G' \leq \Omega_r(G) \leq Z(G)$ which is not possible, as $\mathrm{cl}(G) = 3$. Therefore $\exp(G') = 3^{r+1}$. □

Theorem 2.5. *Every 2-generator $p\mathcal{E}$-group is either abelian or isomorphic to Q_8.*

Proof. Suppose that $G = \langle a, b\rangle$ is a $p\mathcal{E}$-group and that $\exp(G/G') = n = p^r$. Then $a^n, b^n \in G' = \langle [a, b]\rangle$ and since G' is a cyclic p-group, we have $\langle a^n, b^n\rangle = \langle a^n\rangle$ or $\langle b^n\rangle$. Without lose of generality we can suppose that $b^n = a^{ns}$ for some integer s. Then

$$(ba^{-s})^n = [b, a]^{sn(n-1)/2} \tag{1}$$

which by Lemma 2.4 is trivial if p is odd or if $p = 2$ and either $\exp(G') \leq 2^{r-1}$ or $2|s$. In any of these cases we would have that ba^{-s} is in $Z(G)$ and G is abelian. So one can suppose that $p = 2$, that s is odd and that $\exp(G') = 2^r$. In this case the equality (1) implies that $(ba^{-s})^{2n} = 1$ and since G is a $2\mathcal{E}$-group, we have $(ba^{-s})^2 \in Z(G)$. Thus

$$1 = [(ba^{-s})^2, a] = [ba^{-s}, a]^2 = [b, a]^2.$$

It follows that $r \leq 1$ and so $|G| = |\frac{G}{G'}||G'| \leq 8$ and G is either abelian or $G \cong Q_8$ or the dihedral group D_8 of order 8. But D_8 is not a $2\mathcal{E}$-group, since there are elements of order 2 in D_8 which are not central. On the other hand Q_8 is a $2\mathcal{E}$-group, since $\exp(Q_8/Q_8') = 2$ and the only element of order 2 in Q_8 is central. This completes the proof. □

Lemma 2.6. *Let G be a $p\mathcal{E}$-group and let $\exp(\frac{G}{G'}) = p^r$. Then $Z_2(G)^{p^r} = Z(G) \cap G^{p^r}$. In particular if $\mathrm{cl}(G) = 3$, then $\exp(\frac{Z_2(G)}{Z(G)}) = 3^r$.*

Proof. By [11, Theorem 12.3.6], we may assume that $p = 3$. Let $x \in Z_2(G)^{3^r}$. By Lemma 2.3, $x = y^{3^r}$ for some $y \in Z_2(G)$. Since $G^{3^r} \leq G' \leq Z_2(G)$, we have

$$[x, g] = [y^{3^r}, g] = [y, g^{3^r}] = 1$$

for all $g \in G$. This implies that $x \in Z(G)$.
Now assume that $x \in G^{3^r} \cap Z(G)$. Then $x = y^{3^r}$ for some $y \in G$ and so

$$1 = [x, g] = [y^{3^r}, g] = [y, g]^{3^r}$$

for all $g \in G$. Then $[y,g] \in \Omega_r(G) \le Z(G)$ which implies that $y \in Z_2(G)$. Hence $x \in Z_2(G)^{3^r}$. This completes the proof of the first part.
If $\mathrm{cl}(G)=3$ and $Z_2(G)^{3^{r-1}} \le Z(G)$, then we have

$$G^{3^r} = (G^3)^{3^{r-1}} \le Z_2(G)^{3^{r-1}} \le Z(G),$$

(note that since $\gamma_3(G)^3 = 1$, $G^3 \le Z_2(G)$). Thus $G^{3^r} \le Z(G)$ which is impossible by Lemma 2.4(ii). Therefore $\exp(\frac{Z_2(G)}{Z(G)}) = 3^r$. □

Remark 2.7. If G is a 2-Engel group, then $G^3G' \le Z_2(G)$. This is because, $\mathrm{cl}(G) \le 3$ and $\gamma_3(G)^3 = 1$ (see [11, Theorem 12.3.6]). Thus if G is a finite 2-Engel 3-group, then we always have that $\Phi(G) \le Z_2(G)$.

Lemma 2.8. *Let G be a 2-Engel group. If $\frac{G}{Z_2(G)}$ is 2-generator, then $\mathrm{cl}(G) \le 2$.*

Proof. If $\frac{G}{Z_2(G)} = \langle aZ_2(G), bZ_2(G)\rangle$, then $G = \langle a, b, Z_2(G)\rangle$. Thus

$$G' = \langle [x,y], \gamma_3(G) \mid x, y \in \{a,b\} \cup Z_2(G)\rangle.$$

Since G is 2-Engel, we have $G' \le Z(G)$. This completes the proof. □

Theorem 2.9. *Every 3-generator $p\mathcal{E}$-group is nilpotent of class at most 2.*

Proof. For a contradiction suppose that $G = \langle x,y,z\rangle$ is a $p\mathcal{E}$-group of class 3. Suppose that $\exp(G/G') = 3^r$. Let $H = (G')^3\gamma_3(G)$. Notice that, by Lemma 2.4, $[H,G] = H^{3^r} = 1$. Modulo H we have that

$$x^{3^r} = [x,y]^{\alpha}[y,z]^{t_1}[z,x]^{\beta}, \;\; y^{3^r} = [x,y]^{\gamma}[y,z]^{\beta'}[z,x]^{t_2}, \;\; z^{3^r} = [x,y]^{t_3}[y,z]^{\alpha'}[z,x]^{\gamma'},$$

for some integers $\alpha, \beta, \gamma, \alpha', \beta', \gamma', t_1, t_2, t_3 \in \{-1,0,1\}$. Since $[x,y,z] \neq 1$, it follows that t_i's must all be zero. Now since $[x^{3^r}, y] = [x, y^{3^r}]$, one can see that $\beta' = -\beta$. Similarly one can deduce that $\gamma' = -\gamma$ and $\alpha' = -\alpha$. Therefore, modulo H we have

$$x^{3^r} = [x,y]^{\alpha}[z,x]^{\beta}, \;\; y^{3^r} = [x,y]^{\gamma}[y,z]^{-\beta} \;\; z^{3^r} = [y,z]^{-\alpha}[z,x]^{-\gamma}.$$

It follows that

$$[x,y]^{3^r} = [x,y,z]^{\beta}, \;\; [z,x]^{3^r} = [x,y,z]^{-\alpha}, \;\; [y,z]^{3^r} = [x,y,z]^{\gamma}.$$

Then $x^{3^{2r}} = y^{3^{2r}} = z^{3^{2r}} = 1$. Now since G is regular, it follows that $G^{3^{2r}} = 1$ that contradicts Lemma 2.4. □

Theorem 2.10. *Let G be a non-abelian 3-generator $p\mathcal{E}$-group, $\exp(\frac{G}{G'}) = p^r$, $\exp(G') = p^t$ and $p > 2$. Then $|G| = p^{3(r+t)}$ and G has the following presentation*

$$\begin{aligned}\langle x,y,z \mid x^{p^{r+t}} = y^{p^{r+t}} = z^{p^{r+t}} = [x^{p^t}, y] = [x^{p^t}, z] = [y^{p^t}, x] = [y^{p^t}, z] = [z^{p^t}, x] = [z^{p^t}, y] = 1,\\ [x,y] = x^{p^r t_{11}} y^{p^r t_{12}} z^{p^r t_{13}}, [x,z] = x^{p^r t_{21}} y^{p^r t_{22}} z^{p^r t_{23}}, [y,z] = x^{p^r t_{31}} y^{p^r t_{32}} z^{p^r t_{33}}\rangle,\end{aligned}$$

where $1 \le t \le r$ and $[t_{ij}] \in GL(3, \mathbb{Z}_{p^t})$. Moreover every group with the above presentation is a $p\mathcal{E}$-group.

Proof. By Theorem 2.9, $\mathrm{cl}(G) = 2$. Since G is 3-generator, there exist elements $a, b, c \in G$ such that $\frac{G}{Z(G)} = \langle aZ(G)\rangle \times \langle bZ(G)\rangle \times \langle cZ(G)\rangle$. Thus, since G' has exponent p^t, we have $|aZ(G)| = |bZ(G)| = p^t$ and $|cZ(G)| = p^s$ for some integers $t, s \geq 0$ with $t \geq s$. We also have $G' = \langle [a,b], [a,c], [b,c]\rangle$, since $\mathrm{cl}(G) = 2$ and $G = \langle a, b, c, Z(G)\rangle$. Since $|aZ(G)| = p^t$ and $|cZ(G)| = p^s$, we have $|[a,b]| \leq p^t$, $|[a,c]| \leq p^s$ and $|[b,c]| \leq p^s$. Therefore $|G'| \leq p^{t+2s}$. Also, since G is regular, $|G : \Omega_r(G)| = |G^{p^r}|$. Then $|G| \leq |\Omega_r(G)||G^{p^r}| \leq |Z(G)||G'|$ and so $|G : Z(G)| \leq |G'|$. Hence $p^{2t+s} \leq p^{t+2s}$ and $t \leq s$. It follows that $s = t$, $|G'| = |\frac{G}{Z(G)}| = p^{3t}$ and $G' = \langle [a,b]\rangle \times \langle [a,c]\rangle \times \langle [b,c]\rangle$. Since G is not abelian, $t \geq 1$. Thus $\frac{G}{G^p Z(G)} \cong C_p \times C_p \times C_p$. This implies that $G = \langle a, b, c\rangle$.
Now, since $G^{p^r} \leq G'$ and $|G'| = |G : Z(G)| \leq |G : \Omega_r(G)| = |G^{p^r}|$, we have $G' = G^{p^r}$. Also we have $G^{p^r} = \langle a^{p^r}, b^{p^r}, c^{p^r}\rangle$ (since $t \leq r$). By Lemma 2.4 $\exp(G) = p^{r+t}$ and since $G' = G^{p^r}$ is an abelian group of order p^{3t} it follows that $G^{p^r} = \langle a^{p^r}\rangle \times \langle b^{p^r}\rangle \times \langle c^{p^r}\rangle$, $t \leq r$ and $|a| = |b| = |c| = p^{r+t}$. Also since $G^{p^t} = \langle a^{p^t}, b^{p^t}, c^{p^t}\rangle$ and $G^{p^r} = \langle a^{p^r}\rangle \times \langle b^{p^r}\rangle \times \langle c^{p^r}\rangle \leq G^{p^t}$, it is not hard to see that $G^{p^t} = \langle a^{p^t}\rangle \times \langle b^{p^t}\rangle \times \langle c^{p^t}\rangle$ and so

$$p^{3r} = |G^{p^t}| \leq |\Omega_r(G)| \leq |Z(G)| = |G : G'| \leq p^{3r}.$$

It follows that $G^{p^t} = \Omega_r(G) = Z(G)$, $|\Omega_t(G)| = |G : G^{p^t}| = |G'|$ and so $G' = \Omega_t(G)$. Thus we have the following information about G:

$$\begin{aligned}
&|G| = p^{3(r+t)}, \exp(G) = p^{r+t}, G = \langle a, b, c\rangle,\\
&|a| = |b| = |c| = p^{r+t}\\
&Z(G) = \Omega_r(G) = G^{p^t} = \langle a^{p^t}\rangle \times \langle b^{p^t}\rangle \times \langle c^{p^t}\rangle\\
&G' = \Omega_t(G) = G^{p^r} = \langle a^{p^r}\rangle \times \langle b^{p^r}\rangle \times \langle c^{p^r}\rangle
\end{aligned}$$

Hence there exists a 3×3 matrix $T = [t_{ij}] \in GL(3, \mathbb{Z}_{p^t})$ such that

$$\begin{aligned}
[a,b] &= a^{p^r t_{11}} b^{p^r t_{12}} c^{p^r t_{13}}\\
[a,c] &= a^{p^r t_{21}} b^{p^r t_{22}} c^{p^r t_{23}}\\
[b,c] &= a^{p^r t_{31}} b^{p^r t_{32}} c^{p^r t_{33}}
\end{aligned}$$

and every element of G can be written as $a^i b^j c^k$ for some $i, j, k \in \mathbb{Z}$ and

$$\begin{aligned}
(a^i b^j c^k)(a^{i'} b^{j'} c^{k'}) &= a^{i+i'-i'jp^r t_{11} - i'kp^r t_{21} - j'kp^r t_{31}}\\
&b^{j+j'-i'jp^r t_{12} - i'kp^r t_{22} - j'kp^r t_{32}} c^{k+k'-i'jp^r t_{13} - i'kp^r t_{23} - j'kp^r t_{33}}
\end{aligned}$$

Now consider $\widetilde{G} = \mathbb{Z}_{p^{r+t}} \times \mathbb{Z}_{p^{r+t}} \times \mathbb{Z}_{p^{r+t}}$ and define the following binary operation on $\widetilde{G}$:

$$\begin{aligned}
(i,j,k)(i',j',k') = \big(&i + i' - i'jp^r t_{11} - i'kp^r t_{21} - j'kp^r t_{31}, j + j' - i'jp^r t_{12} - i'kp^r t_{22} - j'kp^r t_{32},\\
&k + k' - i'jp^r t_{13} - i'kp^r t_{23} - j'kp^r t_{33}\big)
\end{aligned}$$

It is easy to see that $\widetilde{G}$ with this binary operation is a group and $G \cong \widetilde{G}$. Now one can easily see that the group G has the required presentation. $\square$

Theorem 2.11. (*The main result of* [10]) *For p an odd prime, there exists no finite non-abelian 3-generator p-group having an abelian automorphism group.*

Theorem 2.12. *Let G be a non-abelian finite 3-generator pE-group and $p > 2$. Then $\exp(G') < \exp(\frac{G}{G'})$.*

Proof. Suppose, for a contradiction, that $\exp(G') \geq \exp(\frac{G}{G'})$. By Lemma 2.4, $\exp(G') = \exp(\frac{G}{G'}) = p^r$ and by the proof of Theorem 2.10, we have $G = \langle a, b, c\rangle$ and

$$Z(G) = \Omega_r(G) = G^{p^r} = G' = \langle [a, b]\rangle \times \langle [a, c]\rangle \times \langle [b, c]\rangle, |a| = |b| = |c| = p^{2r}, |[a, b]| = |[a, c]| = |[b, c]| = p^r.$$

If we prove that $Aut(G)$ is abelian, then Theorem 2.11 completes the proof.
Let $\alpha \in Aut(G)$. There exist integers i, n, m and an element $w \in G'$ such that $a^\alpha = a^i b^n c^m w$. Since $[a^\alpha, a] = 1$, we have $[b, a]^n [c, a]^m = 1$ and so $n \equiv m \equiv 0 \pmod{p^r}$. Therefore $a^\alpha = a^i w_a$ and similarly $b^\alpha = b^j w_b$, $c^\alpha = c^k w_c$, where $1 \leq i, j, k \leq p^r - 1$ and $w_a, w_b, w_c \in G' = Z(G)$. From $[(ab)^\alpha, ab] = 1$ and $[(ac)^\alpha, ac] = 1$, it follows respectively that $i = j$ and $i = k$. Also from equality $G^{p^r} = G'$, we have $a^{p^r} = [a, b]^t [b, c]^r [a, c]^s$ where t, r and s are integers. Then $(a^\alpha)^{p^r} = [a^\alpha, b^\alpha]^t [b^\alpha, c^\alpha]^r [a^\alpha, c^\alpha]^s$ and we obtain $a^{p^r i} = x^{p^r i^2}$. Therefore $i^2 \equiv i \pmod{p^r}$ and so $i = 1$. Therefore all automorphisms of G are central so that they fix the elements of $G' = Z(G)$. If $\alpha, \beta \in Aut(G)$, then $x^{\alpha\beta} = x^{\beta\alpha}$ for every $x \in \{a, b, c\}$. Hence $Aut(G)$ is abelian which contradicts Theorem 2.11. □

Corollary 2.13. *Let G be a finite 3-generator pE-group and $p > 2$. If $\exp(G) \leq p^2$, then G is abelian.*

Proof. If G is non-abelian, then by Lemma 2.4 we have $\exp(G') = \exp(\frac{G}{G'}) = p$ which contradicts Theorem 2.12. □

Lemma 2.14. *Let G be a $p\mathcal{E}$-group such that $|G| \leq p^5$. Then G is abelian or G is isomorphic to one of the following groups: Q_8, $Q_8 \times C_2$, $Q_8 \times C_2 \times C_2$,*
$\langle x, y, z \mid x^4 = y^4 = [y, z] = 1, x^2 = z^2 = [x, y], (xz)^2 = y^2\rangle$ *or*
$\langle x, y, z \mid x^4 = z^4 = [y, z] = 1, x^2 = y^2 = [x, y], [x, z] = z^2\rangle$

Proof. It can be easily checked by GAP [4] that there are exactly five non-abelian $2\mathcal{E}$-groups, which are the same as listed in the lemma. Thus we may assume that $p > 2$ and G is non-abelian. If $d(G) \geq 4$, then $|Z(G)| \geq |\Omega_1(G)| = |G : G^p| \geq p^4$ (since G is regular) and so $|G : Z(G)| \leq p$, which is impossible. Hence $d(G) = 3$ and so $|G| \geq p^4$ which contradicts Theorem 2.10. □

Remark 2.15. *Suppose that G is a finite p-group such that $\Omega_1(G) \leq Z(G)$. If G has no non-trivial abelian direct factor, then $\Omega_1(G) \leq \Phi(G)$. To see this, let $x \in G$ be of order p and $x \notin \Phi(G)$. Then there exists a maximal subgroup M such that $x \notin M$. Since $x \in Z(G)$, we have $\langle x\rangle \trianglelefteq G$, so that $G = M \times \langle x\rangle$, a contradiction.*

Theorem 2.16. *Let G be a $p\mathcal{E}$-group having no abelian direct factor and $p > 2$. If $|G| = p^7$, then G is abelian.*

Proof. Suppose, for a contradiction, that G is not abelian. If $d(G) \geq 4$, by Remark 2.15, we have

$$|\Phi(G)| \geq |\Omega_1(G)| = |G : G^p| \geq |G : \Phi(G)| \geq p^4$$

and so $|G| \geq p^8$ which is impossible. Therefore, by Lemma 2.5, we have $d(G) = 3$ which is a contradiction by Theorem 2.10. $\square$

Lemma 2.17. *Let G be an E-group and $a \in G$ be such that $\langle aG' \rangle$ is an infinite direct summand of $\frac{G}{G'}$. Then $a \in Z(G)$.*

Proof. By assumption, we have $\frac{G}{G'} = \langle aG' \rangle \oplus \langle XG' \rangle$ for some $X \subseteq G$. Since G is nilpotent, $G' \leq \Phi(G)$ and so $G = \langle a, X \rangle$. Therefore it is enough to show that $[a, x] = 1$ for all $x \in X$.
Let $\pi : G \to \frac{G}{G'}$ be the natural epimorphism and $\psi : \langle aG' \rangle \oplus \langle XG' \rangle \to \langle aG' \rangle$ the projection map on the first component. Now for each $x \in X$ let $\varphi_x : \langle aG' \rangle \to \langle x \rangle$ be the map defined by $a^iG' \mapsto x^i$ for all $i \in \mathbb{Z}$. Since $\langle aG' \rangle \cong \mathbb{Z}$, φ_x is a group homomorphism mapping aG' to x. Thus $\pi\psi\varphi_x$ is an endomorphism of G mapping a to x. Since G is an E-group, we have that $[a, x] = 1$. This completes the proof. $\square$

Theorem 2.18. *Let G be an infinite finitely generated E-group. Then $G = K \times H$, where K is a central torsion-free subgroup of G and H is a finite subgroup of G. In particular if G is infinite and indecomposable then G is infinite cyclic.*

Proof. Since G is infinite and nilpotent, $\frac{G}{G'}$ is an infinite finitely generated group. Thus

$$\frac{G}{G'} = \langle a_1G' \rangle \oplus \cdots \oplus \langle a_nG' \rangle \oplus \langle b_1G' \rangle \oplus \cdots \oplus \langle b_mG' \rangle \qquad (*)$$

for some $a_1, \ldots, a_n, b_1, \ldots, b_m \in G$ such that $\langle a_iG' \rangle$ is infinite, $\langle b_iG' \rangle$ is finite and $G = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$. By Lemma 2.17, $K = \langle a_1, \ldots, a_n \rangle \leq Z(G)$. It follows that $G' = H'$, where $H = \langle b_1, \ldots, b_m \rangle$. Thus $\frac{H}{H'}$ is finite and since H is a nilpotent group, H is finite. Since $K \leq Z(G)$ and we have the decomposition $(*)$, K is a torsion-free group. It follows that $G = K \times H$. Now if G is indecomposable, we must have $H = 1$ and $n = 1$. This completes the proof. $\square$

Lemma 2.19. *Let G be a finite nilpotent p-group of class 3. If $|G' : G' \cap Z(G)| = p$, then $|G : Z_2(G)| = p^2$.*

Proof. Suppose $G = \langle a, b, c_1, \ldots, c_r \rangle$ where $G'Z(G) = \langle [a, b] \rangle Z(G)$. By replacing c_i by a suitable $c_i a^{\alpha_i} b^{\beta_i}$ one can assume that $[c_i, a], [c_i, b] \in Z(G)$ for $i = 1, \ldots, r$. We claim that $c_1, \ldots, c_r \in Z_2(G)$. For this it suffices to show that $[c_i, c_j] \in Z(G)$ for $1 \leq i < j \leq r$. Suppose

$$[c_i, c_j] = [a, b]^r z$$

with $z \in Z(G)$. As

$$1 = [a, b, c_k][b, c_k, a][c_k, a, b] = [a, b, c_k],$$

this is clear. Hence $G/Z_2(G) = \langle a, b \rangle Z_2(G)/Z_2(G)$ is of order p^2. $\square$

3. Proofs of the main results

Proof of Theorem 1.1. Let G be a 3-generator E-group. If $\frac{G}{G'}$ is finite, since G is nilpotent, G is finite and G is a direct product of its Sylow subgroups. Every Sylow subgroup of G is endomorphic image of G and so by [9] they are at most 3-generator E-groups. In this case, Theorem 2.9 completes the proof. If $\frac{G}{G'}$ is infinite, then by the fundamental theorem of finitely generated abelian groups, we have $\frac{G}{G'} = \langle aG'\rangle \oplus \langle bG', cG'\rangle$ for some $a, b, c \in G$ such that aG' has infinite order. Thus by Lemma 2.17, $a \in Z(G)$ and since G is 2-Engel, it follows easily that $G' = \langle [b, c]\rangle$ and since G is 2-Engel, $\gamma_3(G) = \gamma_3(\langle b, c\rangle) = 1$. This completes the proof. □

Proof of Theorem 1.2. Suppose, for a contradiction, that G is a finite $3E$-group of the least order subject to the properties $\mathrm{cl}(G) = 3$ and $|G| \leq 3^{10}$. Then G is indecomposable and so $\Omega_1(G) \leq \Phi(G)$, by Remark 2.15. Thus $\Omega_1(G) \leq \Phi(G) \cap Z(G)$. If $d(G) \geq 5$, then $|\frac{G}{\Phi(G)}| \geq 3^5$. Since G is regular and $\Phi(G) = G^3G'$, $|\Phi(G) \cap Z(G)| \geq |\Omega_1(G)| = |G : G^3| \geq 3^5$ and since $\mathrm{cl}(G) = 3$, $\Phi(G) \cap Z(G) \lneqq \Phi(G)$. It follows that $|G| \geq 3^{11}$, a contradiction. Thus Theorem 1.1 implies that $d(G) = 4$. If $|G' : Z(G) \cap G'| = 3$, then by Lemma 2.19, we have $|G : Z_2(G)| = 9$. Therefore $\frac{G}{Z_2(G)}$ is a 2-generator group and by Lemma 2.8, $\mathrm{cl}(G) \leq 2$, a contradiction. Hence $|G' : Z(G) \cap G'| \geq 9$. Since

$$|G| = |G : \Phi(G)||\Phi(G) : \Phi(G) \cap Z(G)||\Phi(G) \cap Z(G)|$$

and

$$|\Phi(G) : \Phi(G) \cap Z(G)| = |G'G^3 : G'G^3 \cap Z(G)| = |Z(G)G'G^3 : Z(G)| = \frac{|Z(G)G'||G^3|}{|Z(G)G' \cap G^3||Z(G)|},$$

we have

$$|G| = |G : \Phi(G)||G' : G' \cap Z(G)||G^3 : G'Z(G) \cap G^3||\Phi(G) \cap Z(G)| \geq 3^{10}.$$

Thus $|G| = 3^{10}$, $|\Omega_1(G)| = |\Phi(G) \cap Z(G)| = 3^4$, $|G' : Z(G) \cap G'| = 9$ and $G^3 \leq G'Z(G)$. Since $|G : G^3| = |\Omega_1(G)|$, $G^3 = \Phi(G) \leq Z_2(G)$, $\Phi(G)$ is an abelian group of order 3^6 and $d(\Phi(G)) = 4$. Hence

$$\Phi(G) \cong C_{27} \times C_3 \times C_3 \times C_3 \ \text{ or } \ C_9 \times C_9 \times C_3 \times C_3.$$

Also we have $G'^9 = [G^3, G]^3 \leq (\gamma_3(G))^3 = 1$ and Lemma 2.4(ii) yields that $\exp(\frac{G}{G'}) = 3$. Hence by Lemma 2.6, we have $Z_2(G)^3 = \Phi(G) \cap Z(G)$. Now Lemma 2.8 implies that $d(\frac{G}{Z_2(G)}) = 3$ or 4. Then $|Z_2(G)| = 3^6$ or 3^7, which implies that $Z_2(G) = \Phi(G)$ or $|Z_2(G) : \Phi(G)| = 3$. If $Z_2(G) = \Phi(G)$, then $|Z_2(G)^3| = |\Phi(G) \cap Z(G)| = 9$, a contradiction. Thus $|Z_2(G) : \Phi(G)| = 3$. Since $[Z_2(G), \Phi(G)] = 1$, we have $\Phi(G) \leq Z(Z_2(G))$, $Z_2(G)$ is an abelian group and $d(Z_2(G)) = 4$. Hence

$$Z_2(G) \cong C_{81} \times C_3 \times C_3 \times C_3 \ \text{ or } \ C_{27} \times C_9 \times C_3 \times C_3 \ \text{ or } \ C_9 \times C_9 \times C_9 \times C_3.$$

Thus $|Z_2(G)^3| = |\Phi(G) \cap Z(G)| = 27$. This contradiction completes the proof. □

Proof of Theorem 1.3. *i)* Let G be a 2-generator E-group. Suppose first that $\frac{G}{G'}$ is finite. Since G is nilpotent, G is finite and so it is the direct product of its Sylow subgroups. Every Sylow subgroup of G

is also at most 2-generator and an E-group. Now Theorem 2.5 and Remark 2.2 imply that every Sylow subgroup of G is abelian and so G is abelian.
Therefore we may assume that $\frac{G}{G'}$ is infinite. It follows from the fundamental theorem of finitely generated abelian groups, that $\frac{G}{G'} = \langle aG' \rangle \oplus \langle bG' \rangle$ for some $a, b \in G$ such that $\langle aG' \rangle$ is infinite. Since $G' \leq \Phi(G)$, $G = \langle a, b \rangle$ and Lemma 2.17 completes the proof of (i).
$ii)$ Let G be an infinite 3-generator E-group. Since G is infinite and nilpotent, $\frac{G}{G'}$ is an infinite finitely generated 3-generator group. Thus $\frac{G}{G'} = \langle aG' \rangle \oplus \langle bG' \rangle \oplus \langle cG' \rangle$ for some $a, b, c \in G$ such that $\langle aG' \rangle$ is infinite and $G = \langle a, b, c \rangle$. By Lemma 2.17, $a \in Z(G)$. If either $\langle bG' \rangle$ or $\langle cG' \rangle$ is infinite, then Lemma 2.17 implies that G is abelian. Thus we may assume that $\langle bG' \rangle$ and $\langle cG' \rangle$ are both finite. It follows that $G' = \langle [b, c] \rangle \leq H = \langle b, c \rangle$ is finite. Thus $G = \langle a \rangle \times H$, since $a \in Z(H)$ is of infinite order and H is a finite group. Hence H is a 2-generator E-group, as it is a direct factor of G. Now part (i) completes the proof. □

Proof of Theorem 1.4. Suppose, for a contradiction, that G is a non-abelian pE-group of order p^6. We see that non-abelian groups in Lemma 2.14 are not E-groups and so $|G| = p^6$. If $p = 2$, then one can see (e.g., by GAP [4]) that there exist ten $p\mathcal{E}$-groups T. We have checked by the package `AutPGrp` in GAP [4], that for each such a group T, there are $\alpha \in Aut(T)$ and $x \in T$ such that $[x, x^\alpha] \neq 1$ (the automorphism α is in a set of generators given by the package for $Aut(T)$); thus they are not E-groups. Therefore p is odd. Similar by the proof of Theorem 2.16, we have $d(G) = 3$ (since G has no non-trivial abelian direct factor). Then by Theorem 2.10, we have $\exp(G) = p^2$. Hence by Corollary 2.13, the proof is complete. □

Proof of Theorem 1.5. $i)$ Suppose, for a contradiction, that G is a non-abelian pE-group of least order subject to the property $|G| \leq p^7$. Then by Theorem 1.4, $|G| = p^7$. By the choice of G and that every direct factor of an E-group is again an E-group, G has no abelian direct factor. Now Theorem 2.16 completes the proof.
$ii)$ Let

$$G = \langle x, y, z, t \mid y^2 = z^2, [x, y] = [x, t] = y^2, [x, z] = z^2t^2, [y, z] = x^2, [y, t] = [z, t] = [y, z, t] = 1 \rangle.$$

We have $G' = Z(G) = G^2 = \Omega_1(G) = \{g^2 \mid g \in G\}, \exp(G) = 4$ and $|G| = 2^7$. By Package `AutPGrp` of GAP [4], $Aut(G)$ is abelian and so every automorphism of G is central. Then for all $\beta \in Aut(G)$ and all $a \in G$, $[a, a^\beta] = 1$.
Now we prove that every endomorphism α of G which is not an automorphism, maps G into $Z(G)$. We denote by $Ker\alpha$ and $Im\alpha$ the kernel and image of α, respectively. We first show that if $y^2 \in Ker\alpha$, then $Im\alpha \leq Z(G)$. Since $(y^\alpha)^2 = (z^\alpha)^2 = 1$, we have y^α and $z^\alpha \in Z(G)$. Then $(x^2)^\alpha = [y^\alpha, z^\alpha] = 1$ and so $x^\alpha \in Z(G)$. Also $(t^2)^\alpha = (z^2t^2)^\alpha = [x^\alpha, z^\alpha] = 1$ which implies that $t^\alpha \in Z(G)$. Therefore in this case, $Im\alpha \leq Z(G)$. Thus we can assume $y^2 \notin Ker\alpha$.
Since $\alpha \notin Aut(G)$, $Ker\alpha \cap Z(G) \neq 1$. Now it follows from the equality $G' = \Omega_1(G) = Z(G) = \{g^2 \mid g \in G\}$ that there exists an element $g \in G$ such that $1 \neq g^2 \in Ker\alpha$. Then $g^\alpha \in G'$ and $g \notin G'$. Thus

$g = x^i y^j z^k t^l w$, where $0 \le i, j, k, l \le 1, w \in G'$ and at least one of the integers i, j, k, l is nonzero. For all $a \in G$, $[a, g]^\alpha = [a^\alpha, g^\alpha] = 1$ and so $[a, g] \in Ker\alpha$. Since $[t, g] \in Ker\alpha$, $y^{2i} \in Ker\alpha$ and so $i = 0$. Also $[y, g] \in Ker\alpha$, implies that $x^{2k} \in Ker\alpha$. If $k = 1$ then $(x^2)^\alpha = 1$ and so $x^\alpha \in Z(G)$. Therefore $(y^2)^\alpha = [x^\alpha, t^\alpha] = 1$ which is impossible. Therefore $k = 0$. Since $[z, g] \in Ker\alpha$ and $[x, g] \in Ker\alpha$, we have $j = l = 0$ which is a contradiction and the proof is complete.
By a similar proof one can see that the following groups are non-abelian $2E$-groups of order 2^7.

$$\langle x, y, z, t \mid y^2 = z^2t^2, [x, y] = [x, t] = z^2, [x, z] = t^2, [y, z] = x^2, [y, t] = [z, t] = [y, z, t] = 1\rangle.$$

$$\langle x, y, z, t \mid y^2 = z^2, [x, z] = y^2, [x, t] = t^2y^2, [x, y] = t^2, [y, z] = x^2t^2, [y, t] = t^2, [z, t] = [y, z, t] = 1\rangle.$$

These groups are not isomorphic and their automorphism groups are abelian and every endomorphism which is not an automorphism maps the group into its center. $\square$

Acknowledgment. The authors are grateful to the referee for his/her valuable comments which caused the paper to be shorter.

References

[1] A. Caranti, Finite p-groups of exponent p^2 in which each element commutes with its endomorphic images, J. Algebra **97** (1985) 1-13.
[2] A. Caranti, S. Franciosi, F. de Giovanni, Some examples of infinite groups in which each element commutes with its endomorphic images, Group theoy, Proc. Conf., Brixen/Italy (1986), Lect. Notes Math. **1281** (1987) 9-17.
[3] R. Faudree, Groups in which each element commutes with its endomorphic images, Proc. Amer. Math. Soc. **27** (1971) 236-240.
[4] The GAP Group, GAP-Groups, Algorithms, and Programming, Version 4.4; 2005, (http://www.gap-system.org).
[5] The Kourovka Notebook, Unsolved Problems in Group Theory, 15th augm. ed., Novosibirisk, 2002.
[6] F. W. Levi, Groups in which the commutator operation satisfies certain algebraic condition, J. Indian Math. Soc. **6** (1942) 87-97.
[7] J. J. Malone, A non-abelian 2-group whose endomorphisms generate a ring, and other examples of E-groups, Proc. Edinburgh Math. Soc. **23** (1980) 57-59.
[8] J. J. Malone, Endomorphism near-rings through the ages, Kluwer Academic Publishers. Math. Appl., Dordr. **336** (1995) 31-43.
[9] J. J. Malone, More on groups in which each element commutes with its endomorphic images, Proc. Amer. Math. Soc. **65** (1977) 209-214.
[10] M. Morigi, On the minimal number of generators of finite non-abelian p-groups having an abelian automorphism group, Comm. Algebra **23**, No. 6, (1995) 2045-2065.
[11] D. J. S. Robinson, A course in the theory of groups, Second Edition, Springer-Verlag, New York, 1995.